

ZARISKI CANCELLATION PROBLEM FOR NONCOMMUTATIVE ALGEBRAS

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ABSTRACT. A noncommutative analogue of the Zariski cancellation problem asks whether $A[x] \cong B[x]$ implies $A \cong B$ when A and B are noncommutative algebras. We resolve this affirmatively in the case when A is a noncommutative finitely generated domain over the complex field of Gelfand-Kirillov dimension two. In addition, we resolve the Zariski cancellation problem for several classes of Artin-Schelter regular algebras of higher Gelfand-Kirillov dimension.

0. INTRODUCTION

Kraft said in his 1995 survey [Kr] that “*there is no doubt that complex affine n -space $\mathbb{A}^n = \mathbb{A}_{\mathbb{C}}^n$ is one of the basic objects in algebraic geometry. It is therefore surprising how little is known about its geometry and its symmetries. Although there has been some remarkable progress in the last few years, many basic problems remain open.*” His remark still applies even today—20 years later. Let us start with one of the famous questions in commutative affine geometry. Throughout the introduction, we let k be an algebraically closed field of characteristic zero (except for some results mentioned below).

Question 0.1 (Zariski Cancellation Problem). Does an isomorphism $Y \times \mathbb{A} \cong \mathbb{A}^{n+1}$ imply that Y is isomorphic to \mathbb{A}^n ? Or equivalently, does an isomorphism $B[t] \cong k[t_1, \dots, t_{n+1}]$ of algebras implies that B is isomorphic to $k[t_1, \dots, t_n]$?

For simplicity, let **ZCP** denote the Zariski Cancellation Problem. An algebra A is called *cancellative* if $A[t] \cong B[t]$ for any algebra B implies that $A \cong B$. So the **ZCP** asks if the commutative polynomial ring $k[x_1, \dots, x_n]$ is cancellative. Recall that $k[x_1]$ is cancellative by a result of Abhyankar-Eakin-Heinzer [AEH], $k[x_1, x_2]$ is cancellative by Fujita [Fu] and Miyanishi-Sugie [MS] in characteristic zero and by Russell [Ru] in positive characteristic. The **ZCP** was open for many years. In 2013, a remarkable development was made by Gupta [Gu1, Gu2] who completely settled the **ZCP** negatively in positive characteristic for $n \geq 3$. The **ZCP** in characteristic zero remains open for $n \geq 3$. We give a list of open questions and problems that are closely related to the **ZCP**.

Question 0.2. For the following, let k^\times be $k \setminus \{0\}$.

(ChP:=Characterization Problem) Find an algebro-geometric characterization of \mathbb{A}^n .

(EP:=Embedding Problem) Is every closed embedding $\mathbb{A}^a \hookrightarrow \mathbb{A}^{a+n}$ equivalent to the standard embedding?

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(AP:=Automorphism Problem) Describe the group of polynomial automorphisms of \mathbb{A}^n .

(LP:=Linearization Problem) Is every automorphism of \mathbb{A}^n of finite order linearizable?

(JC:=Jacobian Conjecture) Is every polynomial morphism $\phi : \mathbb{A}^n \rightarrow \mathbb{A}^n$ with $\det J\phi \in k^\times$ an isomorphism?

There are some known relationship between these problems. For example, a positive solution of the **LP** would imply a positive solution of the **ZCP**. When $n \leq 2$, most of these questions (except for the **JC**) were resolved and there is a diagram of implications

$$\mathbf{EP} \implies \mathbf{AP} \implies \mathbf{LP} \implies \mathbf{ZCP}$$

along with a possible “missing link” $\mathbf{JC} \implies \mathbf{ZCP}$ (see [vdE]). Note that the **EP** (in dimension 2) was solved by Abyhyankar-Moh [AM] and Suzuki [Su]. Gupta’s work [Gu1, Gu2] would suggest a negative solution to the **ZCP**, even in characteristic zero. If the “missing link” could be established and if the **ZCP** had a negative solution, then the **JC** could be settled negatively. Many authors have been working on these questions—see the references in [Kr, vdE].

Some naive and direct translations of these questions into the noncommutative setting are easily seen to have negative solutions. So it is important to carefully formulate noncommutative versions of these questions and to understand for which classes of (commutative or noncommutative) algebras these questions have positive or negative answers. Hopefully new ideas will emerge via the study of the noncommutative versions of these open questions. In this paper we mainly consider the following noncommutative formulation of the **ZCP**.

Question 0.3. Let A be a noncommutative noetherian Artin-Schelter regular algebra [AS]. When is A cancellative?

Since Artin-Schelter regular algebras are considered as a noncommutative generalization of the commutative polynomial ring, the above question can be viewed as a noncommutative version of **ZCP**.

In this paper we present two ideas to deal with the **ZCP** for some families of noncommutative algebras. One is to use the *Makar-Limanov invariant* and the other is to use *discriminants*.

Let us first review the Makar-Limanov invariant. Let A be an algebra and let $\text{LND}(A)$ be the collection of locally nilpotent k -derivations of A . The *Makar-Limanov invariant* of A is defined to be

$$\text{ML}(A) = \bigcap_{\delta \in \text{LND}(A)} \ker(\delta).$$

The Makar-Limanov invariant was originally introduced by Makar-Limanov [Ma1] and has become a very useful invariant in commutative algebra. We say that A is *LND-rigid* if $\text{ML}(A) = A$, or equivalently if $\text{LND}(A) = \{0\}$. One of our main results (see Theorem 3.6 for the precise statement and proof) is the following, which shows that rigidity controls cancellation.

Theorem 0.4. *Let A be a finitely generated domain of finite Gelfand-Kirillov dimension. If A is LND-rigid, then A is cancellative.*

By the above theorem, we would like to show that various classes of noncommutative algebras are LND-rigid. Here is one of the consequences [Corollary 3.7].

Theorem 0.5. *Let A be a finitely generated domain of Gelfand-Kirillov dimension two. If A is not commutative, then A is cancellative.*

By [AEH], every commutative domain of Gelfand-Kirillov dimension (or GK-dimension, for short) one is cancellative. By [Da, Fi] there are commutative domains of GK-dimension two that are not cancellative. Theorem 0.5 ensures that every **non**-commutative domain of GK-dimension two is cancellative. Crachiola [Cr] showed that commutative UFDs of GK-dimension two are always cancellative.

Next let us talk about the discriminant method. The discriminant method was introduced in [CPWZ1, CPWZ2] to answer the **AP** for a class of noncommutative algebras. The definition of the discriminant in the noncommutative setting will be reviewed in Section 4. Suppose that A is finitely generated by $Y = \bigoplus_{i=1}^d kx_i$ as an algebra over a base commutative ring k . An element $f \in A$ is called *effective*, if for every testing \mathbb{N} -filtered k -algebra T with $\text{gr } T := \bigoplus F_i T / F_{i-1} T$ being an \mathbb{N} -graded domain, and for every testing subset $\{y_1, \dots, y_d\} \subset T$ satisfying (a) it is linearly independent in the quotient k -module $T/k_1 T$ and (b) some y_i is not in $F_0 T$, there is a presentation of f of the form $f(x_1, \dots, x_d)$ when lifted in the free algebra $k\langle x_1, \dots, x_d \rangle$ such that $f(y_1, \dots, y_d)$ is either zero or not in $F_0 T$. For example, any monomial $x_1^{a_1} \cdots x_d^{a_d}$, for some positive integers a_1, \dots, a_d , is effective. Note that there are non-monomial effective discriminants (see Examples 5.5 and 5.6). Here is one of our main results by using the discriminant, which provides a uniform way of showing the rigidity for some classes of noncommutative algebras.

Theorem 0.6. *Suppose that A is a domain which is a finitely generated module over its affine center C and that the discriminant $d(A/C)$ is effective. Then A is cancellative.*

The above theorem does not solve the original **ZCP** as, when A is commutative, the discriminant over its center is trivial and not effective. However, Theorem 0.6 applies to a large family of noncommutative algebras. One can check, for example, that the skew polynomial ring $k_q[x_1, \dots, x_n]$ where n is even and $1 \neq q$ is a root of unity has effective discriminant. Then, by Theorem 0.6, $k_q[x_1, \dots, x_n]$ is cancellative. The next result shows a connection between the noncommutative **ZCP** and the noncommutative **AP**. Let C denote the center of the algebra A and we refer to Definition 4.4 for the definition of “dominating”. We have the following result (see Theorem 5.7 for an expanded version).

Theorem 0.7. *Let A be a skew polynomial ring $k_{p_{ij}}[x_1, \dots, x_n]$ where each p_{ij} is a root of unity. The following are equivalent.*

- (1) *The full automorphism group $\text{Aut}(A)$ is affine [CPWZ1, Definition 2.5].*
- (2) *The discriminant $d(A/C)$ is dominating.*
- (3) *The discriminant $d(A/C)$ is effective.*
- (4) *A is LND-rigid.*

Consequently, under any of these equivalent conditions, A is cancellative.

In general, by using the Makar-Limanov invariant and Theorem 0.4, we show that if $d(A/C)$ is dominating, then A is cancellative, see Theorem 4.6(2). As an example, we have the following result.

Theorem 0.8. *Let A be any finite tensor product of the algebras of the form*

- (a) $k_p[x_1, \dots, x_n]$ where $1 \neq p \in k^\times$ and n is even;
- (b) $k\langle x, y \rangle / (x^2y - yx^2, y^2x + xy^2)$;
- (c) $k\langle x, y \rangle / (yx - qxy - 1)$ where $1 \neq q \in k^\times$.

Then A is LND-rigid. As a consequence, A is cancellative.

Remark 0.9. Suppose that n is odd and that $q \neq 1$ is a root of unity. It is an open question whether $k_q[x_1, \dots, x_n]$ is cancellative. There are two results related to this.

- (1) The following weak cancellative property holds as a consequence of [BZ, Theorem 3.1]: Let B be a connected graded algebra generated in degree one. If $k_q[x_1, \dots, x_n][t] \cong B[t]$ as algebras, then $k_q[x_1, \dots, x_n] \cong B$ as graded algebras.
- (2) A result of [CYZ2] says that Veronese subrings of $k_q[x_1, \dots, x_n]^{(v)}$ is cancellative when m and v are not coprime, where m is the order of q .

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1. TRIVIAL CENTER VS. CANCELLATION

Throughout the rest of the paper let k be a base commutative domain. Sometimes we further assume that k is a field. Everything is taken over k , for example, \otimes stands for \otimes_k . We sometimes consider k -flat algebras. If k is a field, then every k -module is flat. First we recall the definition of cancellative.

Definition 1.1. Let A be an algebra.

- (a) We call A *cancellative* if $A[t] \cong B[t]$ for some algebra B implies that $A \cong B$.
- (b) We call A *strongly cancellative* if, for any $d \geq 1$, $A[t_1, \dots, t_d] \cong B[t_1, \dots, t_d]$ for some algebra B implies that $A \cong B$.
- (c) We call A *universally cancellative* if, for any k -flat finitely generated commutative domain R such that $R/I = k$ for some ideal $I \subset R$ and any k -algebra B , $A \otimes R \cong B \otimes R$ implies that $A \cong B$.

Remark 1.2. By definition, it is easy to see that

$$\text{universally cancellative} \implies \text{strongly cancellative} \implies \text{cancellative}.$$

But, it is not obvious to us whether any two of them are equivalent.

We have an immediate observation for the noncommutative cancellation problem. Let $C(A)$ denote the center of A .

Proposition 1.3. *Let k be a field and A be an algebra with center $C(A) = k$. Then A is universally cancellative.*

Proof. Let R be any affine commutative domain such that $R/I = k$ for some ideal $I \subset R$ and suppose that $\phi : A \otimes R \rightarrow B \otimes R$ is an algebra isomorphism for some algebra B . Since $C(A) = k$, we have $C(A \otimes R) = R$. Since $C(B \otimes R) = C(B) \otimes R$ and since ϕ induces an isomorphism between the centers, we have

$$(E1.3.1) \quad R \cong C(B) \otimes R.$$

Consequently, $C(B)$ is a commutative domain. By considering the GK-dimension of both sides of (E1.3.1), one sees that $\text{GKdim } C(B) = 0$, when regarded as a

k -algebra. Hence $C(B)$ is a field. Since there is an ideal I such that $R/I = k$, $C(B) = k$. Consequently, $C(B \otimes R) = R$. Now the induced map ϕ is an isomorphism between $C(A \otimes R) = R$ to $C(B \otimes R) = R$, so we have $R/\phi(I) = k$. Finally, ϕ induces an automorphism from $A \cong A \otimes R/(I) \cong B \otimes R/(\phi(I)) \cong B$. \square

There are easy consequences.

Example 1.4. We have the following results.

- (1) Let k be a field of characteristic zero and A_n the n th Weyl algebra. Then $C(A_n) = k$. So A_n is universally cancellative.
- (2) Let k be a field and $q \in k^\times$. Let $k_q[x_1, \dots, x_n]$ be the skew polynomial ring generated by x_1, \dots, x_n subject to the relations $x_j x_i = q x_i x_j$ for all $1 \leq i < j \leq n$. If $n \geq 2$ and q is not a root of unity, then $C(A) = k$. So A is universally cancellative.

2. HIGHER DERIVATIONS AND MAKAR-LIMANOV INVARIANT

The Makar-Limanov invariant is a very useful invariant to deal with the cancellation problem. We will also use a modified version of Makar-Limanov invariant to better control the cancellation in positive characteristic. Given a k -algebra A , let $\text{Der}(A)$ denote the collection of k -derivations of A and $\text{LND}(A)$ denote the collection of locally nilpotent k -derivations of A .

For a sequence of k -linear endomorphisms $\partial := \{\partial_i\}_{i \geq 0}$ of A (satisfying certain finiteness conditions) and for any $c \in k$, define

$$(E2.0.1) \quad G_{c\partial} : A \rightarrow A \quad \text{by} \quad a \rightarrow \sum_{i=0}^{\infty} c^i \partial_i(a)$$

and

$$(E2.0.2) \quad G_{\partial,t} : A[t] \rightarrow A[t] \quad \text{by} \quad a \rightarrow \sum_{i=0}^{\infty} \partial_i(a) t^i, t \rightarrow t,$$

for all $a \in A$.

Definition 2.1. Let A be an algebra.

- (1) A *higher derivation* (or *Hasse-Schmidt derivation*) [HS] on A is a sequence of k -linear endomorphisms $\partial := (\partial_i)_{i \geq 0}$ such that:

$$\partial_0 = id_A, \quad \text{and} \quad \partial_n(ab) = \sum_{i=0}^n \partial_i(a) \partial_{n-i}(b)$$

for all $a, b \in A$ and for all $n \geq 0$. The collection of higher derivations is denoted by $\text{Der}^H(A)$.

- (2) A higher derivation is called *iterative* if $\partial_i \partial_j = \binom{i+j}{i} \partial_{i+j}$ for all $i, j \geq 0$.
 - (3) A higher derivation is called *locally nilpotent* if
 - (a) for all $a \in A$ there exists $n \geq 0$ such that $\partial_i(a) = 0$ for all $i \geq n$,
 - (b) the map $G_{\partial,t}$ defined in (E2.0.2) is an algebra automorphism of $A[t]$.
- The collection of locally nilpotent higher derivations is denoted by $\text{LND}^H(A)$ and the collection of locally nilpotent iterative higher derivations is denoted by $\text{LND}^I(A)$.

- (4) For any $\partial \in \text{Der}^H(A)$, then the kernel of ∂ is defined to be

$$\ker \partial = \bigcap_{i \geq 1} \ker \partial_i.$$

Given a higher derivation $\partial = (\partial_i)_{i \geq 0}$, ∂_1 is necessarily a derivation of A . Hence there is a map $\text{Der}^H(A) \rightarrow \text{Der}(A)$ by sending ∂ to ∂_1 . In characteristic 0, the only iterative higher derivation $\partial = (\partial_i)$ on A such that $\partial_1 = \delta$ is given by:

$$(E2.1.1) \quad \partial_n = \frac{\delta^n}{n!}$$

for all $n \geq 0$. This iterative higher derivation is called the canonical higher derivation associated to δ . In this case, we have a map $\text{Der}(A) \rightarrow \text{Der}^H(A)$ sending δ to (∂_i) as defined by (E2.1.1). Hence the map $\text{Der}(A) \rightarrow \text{Der}^H(A)$ is the right inverse of the map $\text{Der}^H(A) \rightarrow \text{Der}(A)$. The following lemma is easy.

Lemma 2.2. *Let $\partial := (\partial_i)_{i \geq 0}$ be a higher derivation.*

- (1) *Suppose ∂ is locally nilpotent. For any $c \in k$, $G_{c\partial}$ is an algebra automorphism of A .*
- (2) *If ∂ is iterative and satisfies Definition 2.1(3a), then $G_{\partial,t}$ is an algebra automorphism of A . As a consequence, ∂ is locally nilpotent.*
- (3) *If $G : A[t] \rightarrow A[t]$ be a $k[t]$ -algebra automorphism and if $G(a) \equiv a \pmod{t}$ for all $a \in A$, then $G = G_{\partial,t}$ for some $\partial \in \text{LND}^H(A)$.*

Proof. (3) Write $G(a) = \sum_{i \geq 0} \partial_i(a)t^i$ for all $a \in A$. Then it is easy to show that $\partial := (\partial_i)$ is in $\text{LND}^H(A)$. \square

We now recall the definition of the Makar-Limanov invariant.

Definition 2.3. Let A be an algebra over k . Let $*$ be either blank, or H , or I .

- (1) The *Makar-Limanov * invariant* [Ma1] of A is defined to be

$$(E2.3.1) \quad \text{ML}^*(A) = \bigcap_{\delta \in \text{LND}^*(A)} \ker(\delta).$$

This means that we have original $\text{ML}(A)$, as well as, $\text{ML}^H(A)$ and $\text{ML}^I(A)$.

- (2) We say that A is *LND * -rigid* if $\text{ML}^*(A) = A$, or $\text{LND}^*(A) = \{0\}$.
- (3) A is called *strongly LND * -rigid* if $\text{ML}^*(A[t_1, \dots, t_d]) = A$, for all $d \geq 1$.

Remark 2.4. The following hold.

- (a) If k contains \mathbb{Q} , then the induced map $\text{LND}^H(A) \rightarrow \text{LND}(A)$ is surjective and the induced map $\text{LND}(A) \rightarrow \text{LND}^H(A)$ is injective. As a consequence, $\text{ML}^H(A) \subset \text{ML}(A)$. In particular, if A is LND H -rigid, then it is LND-rigid.
- (b) Suppose k contains \mathbb{Q} . Since $\text{LND}(A) = \text{LND}^I(A)$, we have $\text{ML}(A) = \text{ML}^I(A)$.
- (c) If k contains \mathbb{Q} , it is not obvious to us whether $\text{ML}^H(A) = \text{ML}(A)$ in general. In particular, we don't know if LND-rigidity is equivalent to LND H -rigidity.
- (d) If the prime field of k is finite, then there are examples so that

$$\text{ML}^I(A) = \text{ML}^H(A) \supsetneq \text{ML}(A).$$

In particular, the LND H -rigidity is not equivalent to the LND-rigidity. In fact, [CPWZ1, Example 3.9] is such an example.

Example 2.5. Suppose k contains \mathbb{Z} . Define $\partial_n = \frac{1}{n!}(\frac{d}{dt})^n$. Then $\partial := (\partial_n)$ is in $\text{LND}^I(k[t])$ and $\text{LND}^H(k[t])$. One sees that $\text{ML}(k[t]) = k = \text{ML}^I(k[t]) = \text{ML}^H(k[t])$. A similar result holds for $k[t_1, \dots, t_d]$.

Remark 2.6. Suppose A contains \mathbb{Z} . Let $*$ be either blank, or H or I .

- (1) It is clear that $\text{ML}^*(A[t_1, \dots, t_d]) \subseteq \text{ML}^*(A)$, but, it is not obvious to us whether $\text{ML}^*(A[t_1, \dots, t_d]) = \text{ML}^*(A)$.
- (2) Makar-Limanov made the following conjecture in [Ma2]: If A is a commutative domain over a field of characteristic zero, then $\text{ML}(A[t_1, \dots, t_d]) = \text{ML}(A)$. And he proved that the conjecture holds when $\text{GKdim } A = 1$ [Ma2].

3. RIGIDITY CONTROLS CANCELLATION

We shall investigate the relationship between LND-rigidity (respectively, strong LND-rigidity) and cancellation (respectively, strong cancellation).

We need the following lemma which is [KL, Proposition 3.11]. If A is an algebra over a commutative base ring k (which is not a field in general), then the Gelfand-Kirillov dimension (or GK-dimension, for short) of A is defined to be

$$\text{GKdim } A = \sup_V \left(\overline{\lim}_{n \rightarrow \infty} \log_n \text{rank}_k(V^n) \right)$$

where V varies over all finitely generated k -submodules of A .

Lemma 3.1. *Let k be a commutative domain. Let A be a k -algebra and R be an affine commutative k -algebra.*

- (1) $\text{GKdim } A = \text{GKdim}_Q A \otimes Q$ where Q is the field of fractions of k . In particular, if A is affine and commutative, $\text{GKdim } A$ is an integer.
- (2) $\text{GKdim } A \otimes R = \text{GKdim } A + \text{GKdim } R$.

Proof. Left to the reader. □

Lemma 3.2. *Let $Y := \bigoplus_{i=0}^{\infty} Y_i$ be an \mathbb{N} -graded domain. If Z is a subalgebra of Y containing Y_0 such that $\text{GKdim } Z = \text{GKdim } Y_0 < \infty$, then $Z = Y_0$.*

Proof. Let X denote the subalgebra Y_0 . Suppose Z strictly contains X as a subalgebra. Since Y is a graded algebra, Z is an \mathbb{N} -filtered algebra with $F_0 Z = X$. By (a non-field version of) [KL, Lemma 6.5], $\text{GKdim } Z \geq \text{GKdim } \text{gr } Z$. Since $\text{gr } Z$ is an \mathbb{N} -graded sub-domain of Y that strictly contains X as the degree zero part of $\text{gr } Z$, one can easily see that $\text{GKdim } \text{gr } Z \geq \text{GKdim}(\text{gr } Z)_0 + 1 = \text{GKdim } X + 1$. Combining these inequalities, one obtains that $\text{GKdim } Z \geq \text{GKdim } X + 1$. This contradicts the hypothesis that $\text{GKdim } Z = \text{GKdim } X$. Therefore $Z = X$. □

It is well-known that a domain of finite GK-dimension is an Ore domain. Here is the main result of this section.

Theorem 3.3. *Let A be a finitely generated domain of finite GK-dimension. Let $*$ be either blank, or H , or I . When $*$ is blank we further assume A contains \mathbb{Z} .*

- (1) *If A is strongly LND^* -rigid, then A is strongly cancellative.*
- (2) *If $\text{ML}^*(A[t]) = A$, then A is cancellative.*

Proof. We prove (1) and note that the proof of (2) is similar.

Let $\phi : A[t_1, \dots, t_d] \rightarrow B[t_1, \dots, t_d]$ be an isomorphism for some algebra B . Since A has finite GK-dimension, so is B and $\text{GKdim } B = \text{GKdim } A$ [Lemma 3.1(b)]. For

each i , let $\partial_i := \frac{\partial}{\partial t_i}$ when $*$ is blank and $\partial_i := \{\frac{1}{n!}(\frac{\partial}{\partial t_i})^n\}_{n=0}^\infty$ when $*$ is either H or I . We have that $\text{ML}^*(B[t_1, \dots, t_d])$ is contained in B since differentiation with respect to t_i , namely ∂_i , gives rise to a locally nilpotent derivation of $B[t_1, \dots, t_d]$ and the intersection of the kernels of these maps is exactly B . On the other hand, we have that $\text{ML}^*(A[t_1, \dots, t_d]) = A$ by hypothesis. If ∂ is a locally nilpotent derivation of B then $\partial \circ \phi$ is a locally nilpotent derivation of B and similarly if ∂' is a locally nilpotent derivation of A then $\partial' \circ \phi^{-1}$ is a locally nilpotent derivation of A . Thus ϕ induces an isomorphism between $\text{ML}^*(A[t_1, \dots, t_d])$ and $\text{ML}^*(B[t_1, \dots, t_d])$. In particular ϕ maps A into B . Let $Y = A[t_1, \dots, t_d]$ with $\deg t_i = 1$ and $Y_0 = A$ and $Z = \phi^{-1}(B)$. Then Lemma 3.2 implies that $\phi^{-1}(B) = A$. So A and B are isomorphic. The result follows. \square

For the rest of this section we give some corollaries. We begin with a well-known result (see [BS, Lemma 3.2] or [Ba, Lemma 2.1] for related results). If A is an Ore domain, let $Q(A)$ denote the fraction division ring of A .

Lemma 3.4. *Let A be an Ore domain containing \mathbb{Z} . Suppose that A is endowed with a nonzero locally nilpotent derivation δ . Then the following hold.*

- (1) *A is embedded in the Ore extension $E[x; \delta_0]$ and $E[x; \delta_0]$ is embedded in $Q(A)$, where $E = \{a \in Q(A) \mid \delta(a) = 0\}$ and δ_0 is a derivation of E .*
- (2) *$Q(A) = Q(E[x; \delta_0])$.*
- (3) *δ can be extended to a locally nilpotent derivation of $E[x; \delta_0]$ by declaring that $\delta(E) = 0$ and $\delta(x) = 1$.*

Proof. (1) Let E denote the kernel of the unique extension of δ to $Q(A)$. Then E is a division subalgebra of $Q(A)$. Since δ is nonzero and locally nilpotent, we can find $x \in A \setminus E$ such that $\delta(x) \in E$. By replacing x by αx for some $\alpha \in E$ we may assume that $\delta(x) = 1$. Now for every $a \in E$ we have $\delta([x, a]) = [\delta(x), a] = [1, a] = 0$. Thus $[x, a] \in E$ for all $a \in E$. In particular, $[x, -]$ induces a derivation δ_0 of E .

Let

$$W = \{a \in Q(A) \mid \delta^n(a) = 0, \text{ for some } n \geq 0\}.$$

We claim that W is a subset of the subalgebra of $Q(A)$ generated by E and x . Since $[x, E] \subseteq E$, we have that this subalgebra is just

$$\sum_{i \geq 0} Ex^i.$$

To see the claim, we let $a \in W$. Then there is some smallest n for which $\delta^n(a) = 0$. We prove the claim by induction on n . When $n = 0$ we have $a \in E$ and so the result follows. Now suppose that the claim holds whenever $\delta^j(a) = 0$ for some $j < n$ and consider the case that $\delta^n(a) = 0$ but $\delta^j(a) \neq 0$ for $j < n$. Then $\delta^{n-1}(a) = \alpha \in E$ with $\alpha \neq 0$. Since $\delta^{n-1}(\alpha x^{n-1}/(n-1)!) = \alpha$, we see that $\delta^{n-1}(a - \alpha x^{n-1}/(n-1)!) = 0$ and so by the inductive hypothesis $a \in \sum Ex^i$. The claim follows.

It is clear that $\sum Ex^i \subseteq W$. So $W = \sum Ex^i$. Since δ is in $\text{LND}(A)$, $A \subset W$. Thus A embeds in the subalgebra W generated by E and x . Since $[\alpha, x] = \delta_0(\alpha)$ for $\alpha \in E$, we see that W is isomorphic to a homomorphic image of $E[t; \delta_0]$. If W is in fact isomorphic to a proper homomorphic image of $E[t; \delta_0]$, then A embeds in a finitely generated free E -module and since δ is zero on E , we see that it must be zero on the algebra generated by E and x and hence it must be identically zero

on A , a contradiction. Thus we see that A embeds in W which is isomorphic to $E[x; \delta_0]$ as required.

Both (2) and (3) are clear. \square

The following result was proved in [Ma2] in the commutative case.

Lemma 3.5. *Let A be a finitely generated Ore domain over k that contains \mathbb{Z} . If A is LND-rigid, then $\text{ML}(A[x]) = A$.*

Proof. Let $C = \text{ML}(A[x])$. Note that $C \subseteq A$ since differentiation with respect to x gives a locally nilpotent derivation of $A[x]$ and the kernel of this map is exactly A . It suffices to show that $C \supseteq A$. Suppose that there is a locally nilpotent derivation δ of $A[x]$ that does not send A to zero. Suppose a_1, \dots, a_s generate A as a k -algebra. Then $\delta(A) \subseteq A\delta(a_1)A + \dots + A\delta(a_s)A$ and so there exists some smallest $m \geq 0$ such that $\delta(A) \subseteq A + Ax + \dots + Ax^m$. If $m = 0$, then $\delta(A) \subseteq A$. Since A is LND-rigid, $\delta(A) = 0$. This yields a contradiction and therefore $m \geq 1$. We write

$$\delta(a) = \mu(a)x^m + \text{lower degree terms}$$

for some derivation μ of A . We now consider the following three cases.

Case I: $\delta(x) \in A + Ax + \dots + Ax^m$.

In this case we have $\delta(x^i) \subseteq \sum_{n=0}^{i+m-1} Ax^n$ and $\delta(Ax^i) \subseteq \sum_{n=0}^{i+m} Ax^n$ for all i . Thus

$$\delta^2(a) = \mu^2(a)x^{2m} + \text{lower degree terms}.$$

More generally, we see that

$$\delta^j(a) = \mu^j(a)x^{mj} + \text{lower degree terms}.$$

Thus μ is a locally nilpotent derivation and so $\mu(A) = 0$, contradicting the minimality of m . Thus $\delta(A) = 0$ in this case.

Case II: $\delta(x) = bx^{m+1} + \text{lower degree terms}$ for some $b \neq 0$ in A .

Applying δ to the equation $[x, a] = 0$, one sees that b commutes with a , or b is in the center of A . Now we define a new derivation δ' of $A[x]$ by declaring that $\delta'(a) = \mu(a)x^m$ for $a \in A$ and $\delta'(x) = bx^{m+1}$. Then we see that δ' sends Ax^i to Ax^{i+m+1} for every $i \geq 0$. We can view δ' as an associated graded derivation of δ . Since δ is locally nilpotent, δ' is a locally nilpotent derivation of $A[x]$ [CPWZ2, Lemma 4.11]. Applying Lemma 3.4 to the algebra $A[x]$, $A[x]$ embeds in $E[y; \delta_0]$ where δ_0 is a derivation of E . Moreover, δ' extends to a locally nilpotent derivation of $E[y; \delta_0]$ by declaring that $\delta'(E) = 0$ and $\delta'(y) = 1$. Under this embedding $x = p(y)$ for some nonzero polynomial p . Let d denote the degree of this polynomial. Then bx^{m+1} gets sent to $q(y)p(y)^{m+1}$ for some nonzero polynomial $q(y)$. But since $\delta'(x)$ is nonzero, it has degree exactly $d - 1$ and so we have $(m + 1)d + \deg q(y) = d - 1$, which is impossible.

Case III: $\delta(x) = bx^i + \text{lower degree terms}$ for some $b \neq 0$ in A and some $i > m + 1$.

In this case we see that, for any $n \geq 2$,

$$\delta^n(x) = \left\{ \prod_{s=1}^{n-1} ((i-1)s + 1) \right\} b^n x^{(i-1)n+1} + \text{lower degree terms},$$

so δ cannot be locally nilpotent, which contradicts the hypothesis.

Combining these cases, $\delta(A) = 0$. The result follows. \square

We next give the proof of Theorem 0.4.

Theorem 3.6. *Let A be a finitely generated domain containing \mathbb{Z} . Suppose it has finite GK-dimension. If A is LND-rigid, then A is cancellative.*

Proof. Since A is a domain of finite GK-dimension, it is an Ore domain. By Lemma 3.5, $\text{ML}(A[x]) = A$. Then the assertion follows from Theorem 3.3(2). \square

We now prove Theorem 0.5. We say an algebra A is *PI* if it satisfies a polynomial identity.

Corollary 3.7. *Let A be a domain of GK-dimension two over an algebraically closed field k of characteristic zero.*

- (1) *If A is PI and not commutative, then A is LND-rigid. As a consequence, if further A is finitely generated over k , then A is cancellative.*
- (2) *If A is not PI, then A is universally cancellative.*

Proof. (1) If A is not LND-rigid, then there is a nonzero locally nilpotent derivation δ of A . So the kernel of δ is not equal to A . As in Lemma 3.4 let E denote the set of elements $a \in Q(A)$ such that $\delta(a) = 0$. By Lemma 3.4, A embeds in $W := E[x; \delta_0]$ for some derivation δ_0 of E . Since W is a subalgebra $Q(A)$, $Q(A)$ is infinite dimensional as a left and right E -vector space. Hence E has GK-dimension one [Be, Theorem 1.3]. By the Small-Warfield theorem [SW] and Tsen's theorem, every domain of GK-dimension one is commutative, whence E is a field. By Lemma 3.4(3), $W := E[x; \delta_0]$ is a subring of $Q(A)$. Since A is PI, $Q(A)$ and then $E[x; \delta_0]$ are PI. This implies that $\delta_0 = 0$ and W is commutative. So A is commutative, yielding a contradiction. The result follows.

The consequence follows from the main assertion and Theorem 3.6.

- (2) If A is not PI and has GK-dimension two, then, by [SZ, Corollary 2], $C(A) = k$. The assertion follows from Proposition 1.3. \square

Definition 3.8. An Ore domain A is called *birationally affine-ruled* if $Q(A) = D(x)$ for some division algebra D and *birationally Weyl-ruled* if $Q(A) = Q(E[x; \delta_0])$ for some nonzero derivation δ_0 of E .

By Lemma 3.4, if A is endowed a nonzero locally nilpotent derivation, then A is either birationally affine-ruled or birationally Weyl-ruled.

Corollary 3.9. *Let A be a finitely generated PI domain containing \mathbb{Z} with finite GK-dimension. If A is not birationally affine-ruled, then A is LND-rigid and cancellative.*

Proof. By Theorem 3.6, it suffices to show that A is LND-rigid.

If A is not LND-rigid, then A is endowed with a nonzero locally nilpotent derivation. By Lemma 3.4, $A \subset E[x; \delta_0] \subset Q(A)$ where E is a division subring of $Q(A)$. Since A is PI, so are $Q(A)$ and $E[x; \delta_0]$. Then the center of $E[x; \delta_0]$ is not a subring of E . Let $f = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ be a central element in $E[x; \delta_0]$ for some $n \geq 1$ and $a_0 \neq 0$. Since f is central,

$$0 = [x, f] = \sum_{i=0}^n [x, a_i] x^i = \sum_{i=0}^n \delta_0(a_i) x^i,$$

which implies that $\delta_0(a_i) = 0$ for all i . For any $e \in E$,

$$0 = [e, f] = [e, a_n]x^n + \text{lower degree terms},$$

which implies that $[e, a_n] = 0$. Hence, a_n is in the center of $E[x; \delta_0]$. By replacing f by $a_n^{-1}f$, we may assume that $a_n = 1$. By calculation,

$$\begin{aligned} 0 = [e, f] &= ex^n - (ex^n + n\delta_0(e)x^{n-1} + \text{lower degree terms}) \\ &\quad + [e, a_{n-1}]x^{n-1} + \text{lower degree terms} \\ &= (-n\delta_0(e) + [e, a_{n-1}])x^{n-1} + \text{lower degree terms}. \end{aligned}$$

Hence $-n\delta_0(e) + [e, a_{n-1}] = 0$ or $\delta_0(e) = [e, b]$ where $b = \frac{1}{n}a_{n-1}$. Then $E[x, \delta_0] = E[x']$ where $x' = x + b$. So A is birationally affine-ruled, a contradiction. \square

There are many commutative domains that are not birationally affine-ruled, so these are necessarily LND-rigid and cancellative.

4. DISCRIMINANT

We recall the definition of the discriminant in the noncommutative setting and everything in this section is taken from [CPWZ1, CPWZ2]. Let R be a commutative algebra and let B and F be algebras both of which contain R as a subalgebra. In applications, F would be either R or a ring of fractions of R . An R -linear map $\text{tr} : B \rightarrow F$ is called a *trace map* if $\text{tr}(ab) = \text{tr}(ba)$ for all $a, b \in B$.

If B is the $w \times w$ -matrix algebra $M_w(R)$ over R , the internal trace $\text{tr}_{\text{int}} : B \rightarrow R$ is defined to be the usual matrix trace, namely, $\text{tr}_{\text{int}}((r_{ij})) = \sum_{i=1}^w r_{ii}$. Let B be an R -algebra, and suppose that $B_F := B \otimes_R F$ is finitely generated free over F , where F is a localization of R . Then the left multiplication defines a natural embedding of R -algebras $lm : B \rightarrow B_F \rightarrow \text{End}_F(B_F) \cong M_w(F)$, where w is the rank $\text{rk}(B_F/F)$. Then we have a *regular trace*, by composing:

$$\text{tr}_{\text{reg}} : B \xrightarrow{lm} M_w(F) \xrightarrow{\text{tr}_{\text{int}}} F.$$

Usually we use the regular trace map even if other trace functions exist. The following definition is well-known, see Reiner's book [Re]. For any algebra A , let A^\times denote the set of invertible elements in A . If $f, g \in A$ and $f = cg$ for some $c \in A^\times$, then we write $f =_{A^\times} g$.

Definition 4.1. [CPWZ1, Definition 1.3] Let $\text{tr} : B \rightarrow F$ be a trace map and v be a fixed integer. Let $Z := \{z_i\}_{i=1}^v$ be a subset of B .

- (1) The *discriminant* of Z is defined to be

$$d_v(Z : \text{tr}) = \det(\text{tr}(z_i z_j))_{v \times v} \in F.$$

- (2) [Re, Section 10, p. 126]. The *v-discriminant ideal* (or *v-discriminant R-module*) $D_v(B, \text{tr})$ is the R -submodule of F generated by the set of elements $d_v(Z : \text{tr})$ for all $Z = \{z_i\}_{i=1}^v \subset B$.
- (3) Suppose B is an R -algebra which is finitely generated free over R of rank w . If Z is an R -basis of B , the *discriminant* of B over R is defined to be

$$d(B/R) =_{R^\times} d_w(Z : \text{tr}).$$

Note that $d(B/R)$ is well-defined up to a scalar in R^\times [Re, p.66, Exer 4.13].

We refer to the books [AW, Re, St] for the classical definition of the discriminant and its connection with the above definition.

To cover a larger class of algebras that are not free over their center, we need a modified version of the discriminant. Let B be a domain and let $\mathcal{D} := \{d_i\}_{i \in I}$ be a set of elements in B . A normal element $x \in B$ is called a *common divisor* if $d_i = d'_i x$ for some d'_i for all $i \in I$. We say a normal element $x \in B$ is the *greatest common divisor* or *gcd* of \mathcal{D} , denoted by $\gcd \mathcal{D}$, if

- (a) x is a common divisor of \mathcal{D} , and
- (b) for any common divisor y of \mathcal{D} , $x = cy$ for some $c \in B$.

It follows from part (b) that the gcd of any subset $\mathcal{D} \subseteq B$ (if it exists) is unique up to a scalar in B^\times .

Definition 4.2. [CPWZ2, Definition 1.2] Let $\text{tr} : B \rightarrow R$ be a trace map and v any positive integer. Let Z (respectively, Z') denote any v -element subset $\{z_i\}_{i=1}^v$ (respectively, $\{z'_i\}_{i=1}^v$) of B .

- (1) The *discriminant* of the pair (Z, Z') is defined to be

$$d_v(Z, Z' : \text{tr}) = \det(\text{tr}(z_i z'_j))_{v \times v} \in R.$$

- (2) The *modified v -discriminant ideal* $MD_v(B, \text{tr})$ is the ideal of R generated by the set of elements $d_v(Z, Z' : \text{tr})$ for all $Z, Z' \subset B$.
- (3) The *v -discriminant* $d_v(B/R)$ is defined to be the gcd in B (possibly not in R) of elements $d_v(Z, Z' : \text{tr})$ for all $Z, Z' \subset B$. Equivalently, the *v -discriminant* $d_v(B/R)$ is the gcd in B of all elements in $MD_v(B, \text{tr})$.

In Definition 4.2(3), we are taking the gcd in B , not in R . If $d_v(B/R)$ exists, then the ideal $(d_v(B/R))$ of B generated by $d_v(B/R)$ is the smallest principal ideal of B that contains $MD_v(B : \text{tr})B$. If B is an R -algebra which is finitely generated free over R and if $w = \text{rk}(B/R)$, then $MD_w(B : \text{tr})$ equals $D_w(B : \text{tr})$, both of which are generated by a single element $d(B/R)$. In this case it is also true that $d(B/R) =_{B^\times} d_w(B/R)$. If $v > \text{rk}(B/R)$, then $d_v(B/R) = 0$ [CPWZ2, Lemma 1.9(2)].

Some explicit examples of discriminants are given in [CPWZ1, CPWZ2, CYZ1].

One of key lemmas is the following, which suggests that the discriminant controls the automorphisms.

Lemma 4.3. *Let $\phi : A \rightarrow B$ be an isomorphism of algebras that restricts to an isomorphism of central subalgebras $\phi^R : R_A \rightarrow R_B$. Suppose that tr_A (respectively, tr_B) is the regular trace $A \rightarrow R_A$ (resp. $B \rightarrow R_B$) and that the image of tr_A is in R_A (resp. the image of tr_B is in R_B). Let w be any positive integer. Then the following hold:*

- (1) ϕ maps the discriminant ideal $D_w(A, \text{tr}_A)$ to $D_w(B, \text{tr}_B)$;
- (2) if A is a finitely generated free module over R_A , then the discriminant $\phi(d(A/R_A)) =_{R_B^\times} d(B/R_B)$;
- (3) ϕ maps the modified discriminant ideal $MD_w(A, \text{tr}_A)$ to $MD_w(B, \text{tr}_B)$;
- (4) ϕ maps the w -discriminant ideal $d_w(A/R_A)$ to $d_w(B/R_B)$.

Proof. First note that $\phi(\text{tr}_A(x)) = \text{tr}_B(\phi(x))$ for all $x \in A$. The rest follows from this observation. \square

The concept of a dominating element was introduced in [CPWZ1, CPWZ2] to handle the noncommutative **AP**. We now recall this notion.

Let R be an algebra over k . We say R is *connected graded* if $R = k \oplus R_1 \oplus R_2 \oplus \cdots$ and R is *locally finite* if each R_i is finitely generated over k . We now consider filtered rings A . Let Y be a finitely generated free k -submodule of A such that $k \cap Y = \{0\}$. Consider the *standard filtration* defined by $F_n A := (k + Y)^n$ for all $n \geq 0$. Assume that this filtration is exhaustive and that the associated graded ring $\text{gr } A$ is connected graded. For each element $f \in F_n A \setminus F_{n-1} A$, the associated element in $\text{gr } A$ is defined to be $\text{gr } f = f + F_{n-1} A \in (\text{gr}_F A)_n$. The degree of an element $f \in A$, denoted by $\deg f$, is defined to be the degree of $\text{gr } f$.

If $\text{gr } A$ is a domain, then, for any elements $f_1, f_2 \in A$,

$$\deg(f_1 f_2) = \deg f_1 + \deg f_2.$$

If $\text{gr } A$ is a connected graded domain, it is easy to see that $A^\times = k^\times$. If R is any subalgebra of A , then $R^\times \subseteq A^\times = k^\times$.

Suppose now A is generated by elements in $Y = \bigoplus_{i=1}^n kx_i$ of degree 1. A monomial $x_1^{b_1} \cdots x_n^{b_n}$ is said to have degree *component-wise less than* (or, *cwlt*, for short) $x_1^{a_1} \cdots x_n^{a_n}$ if $b_i \leq a_i$ for all i and $b_{i_0} < a_{i_0}$ for some i_0 . We write $f = cx_1^{b_1} \cdots x_n^{b_n} + (\text{cwlt})$ if $f - cx_1^{b_1} \cdots x_n^{b_n}$ is a linear combination of monomials with degree component-wise less than $x_1^{b_1} \cdots x_n^{b_n}$.

Definition 4.4. [CPWZ1, Definition 2.1(2)] Retain the above notation. Suppose that $Y = \bigoplus_{i=1}^n kx_i$ generates A as an algebra.

- (1) An element $f \in A$ is called *locally dominating* if f can be written as $f(x_1, x_2, \dots, x_n)$ such that, for every $g \in \text{Aut}(A)$, one has
 - (a) $\deg f(y_1, \dots, y_n) \geq \deg f$, where $y_i = g(x_i)$ for all $i \leq n$, and
 - (b) $\deg f(y_1, \dots, y_n) > \deg f$ if, further, $\deg y_{i_0} > 1$ for some $i_0 \leq n$.
- (2) Suppose $\text{gr } A$ is a connected graded domain. An element $f \in A$ is called *dominating* if, for every testing N-filtered PI algebra T with $\text{gr } T$ being a connected graded domain, and for every testing subset $\{y_1, \dots, y_n\} \subset T$ that is linearly independent in the quotient k -module $T/F_0 T$, there is a presentation of f of the form $f(x_1, \dots, x_n)$ in the free algebra $k\langle x_1, \dots, x_n \rangle$, such that the following hold: either $f(y_1, \dots, y_n) = 0$, or
 - (a) $\deg f(y_1, \dots, y_n) \geq \deg f$, and
 - (b) $\deg f(y_1, \dots, y_n) > \deg f$ if, further, $\deg y_{i_0} > 1$ for some i_0 .

If $f = x_1^{b_1} \cdots x_n^{b_n} + (\text{cwlt})$ for some $b_1, \dots, b_n \geq 1$, then f is dominating, see the proof of [CPWZ1, Lemma 2.2]. It is easy to check that dominating elements are indeed locally dominating. In the next section we will introduce a notion of effectiveness to deal with noncommutative **ZCP**.

The next is a key lemma. Let B be a commutative algebra. We say that $A \otimes B$ is *A-closed* if, for every $0 \neq f \in A$ and $x, y \in A \otimes B$, the equation $xy = f$ implies that $x, y \in A$ up to units of $A \otimes B$. For example, if B is connected graded and $A \otimes B$ is a domain, then $A \otimes B$ is *A-closed*.

Lemma 4.5. [CPWZ2, Lemma 1.12] *Let B be a k -flat commutative algebra such that $A \otimes B$ is a domain and v be a positive integer.*

- (1) $MD_v(A \otimes B : \text{tr} \otimes B) = MD_v(A : \text{tr}) \otimes B$.
- (2) *Suppose $A \otimes B$ is A-closed. If $d_v(A/C)$ exists, then $d_v(A \otimes B/C \otimes B)$ exists and equals $d_v(A/C)$.*

Now we are ready to state the main result of this section, which is basically [CPWZ1, Lemma 3.3(3)]. The proof is given in the next section.

Theorem 4.6. *Let A be a PI algebra. Suppose that the w -discriminant $d_w(A/C)$ is dominating for some w . Then the following hold.*

- (a) *A is strongly LND^H -rigid.*
- (b) *If A has finite GK-dimension, then A is strongly cancellative.*

The above theorem applies to many algebras including ones listed below.

Example 4.7. It is known that the following algebras have dominating discriminants [CPWZ1].

- (1) $k_q[x_1, \dots, x_n]$ where n is an even number and $1 \neq q$ is a root of unity.
- (2) $k\langle x, y \rangle / (x^2y - yx^2, y^2x + xy^2)$.
- (3) $k\langle x, y \rangle / (yx - qxy - 1)$ where $1 \neq q$ is a root of unity.
- (4) finite tensor product of any algebras of (1,2,3) above [CPWZ1, Lemma 5.4].

By Theorem 4.6(2), these algebras are strongly cancellative.

5. EFFECTIVENESS CONTROLS CANCELLATION

First we introduce the notion of effectiveness that plays an important role in the resolution of noncommutative **ZCP**.

Definition 5.1. Let A be a domain and suppose that $Y = \bigoplus_{i=1}^n kx_i$ generates A as an algebra. An element $f \in A$ is called *effective* if the following conditions hold.

- (1) There is an \mathbb{N} -filtration $\{F_i A\}_{i \geq 0}$ on A such that the associated graded ring $\text{gr } A$ is a domain (one possible filtration is the trivial filtration $F_0 A = A$). With this filtration we define the degree of elements in A , denoted by \deg_A .
- (2) For every testing \mathbb{N} -filtered PI algebra T with $\text{gr } T$ being an \mathbb{N} -graded domain and for every testing subset $\{y_1, \dots, y_n\} \subset T$ satisfying
 - (a) it is linearly independent in the quotient k -module $T/k1_T$, and
 - (b) $\deg y_i \geq \deg x_i$ for all i and $\deg y_{i_0} > \deg x_{i_0}$ for some i_0 ,
there is a presentation of f of the form $f(x_1, \dots, x_n)$ in the free algebra $k\langle x_1, \dots, x_n \rangle$, such that either $f(y_1, \dots, y_n)$ is zero or $\deg_T f(y_1, \dots, y_n) > \deg_A f$.

It is clear that a dominating element is effective by taking the special (e.g., standard) filtration of A . We have already seen that there are many examples (those example given in [CPWZ1, CPWZ2]) of noncommutative algebras whose discriminant is dominating, whence effective. Here is the main result in this section, which is a slight generalization of Theorem 4.6.

Theorem 5.2. *Let A be a PI domain such that the w -discriminant over its center is effective for some w .*

- (1) *A is strongly LND^H -rigid.*
- (2) *Suppose A has finite GK-dimension. Let R be an affine k -flat connected graded commutative domain such that $A \otimes R$ is a domain. If $A \otimes R \cong B \otimes R$ for some algebra B , then $A \cong B$.*
- (3) *Suppose A has finite GK-dimension. Then A is strongly cancellative.*

Proof. (1) Suppose A is generated by $\{x_1, \dots, x_n\}$ as in Definition 5.1. Let $B = k[t_1, \dots, t_d][t]$. By Lemma 4.5(2),

$$d_w(A \otimes B/C \otimes B) = d_w(A/C) =: f,$$

which is effective by hypothesis. Let $\partial \in \text{LND}^H(A[t_1, \dots, t_d])$. By definition, $G := G_{\partial, t} \in \text{Aut}_{k[t]}(A[t_1, \dots, t_d][t])$. Then $G(x_j) = x_j + \sum_{i \geq 1} t^i \partial_i(x_j)$. Set the test algebra $T = A[t_1, \dots, t_d][t]$ where the filtration on T is induced by the filtration on A together with $\deg t_s = 0$ for all $s = 1, \dots, d$ and $\deg t = \alpha > \deg(x_j)$ for all $j = 1, \dots, n$. Now set $y_j = G(x_j) \in T$. Since $\deg x_j < \deg t = \alpha$, we have that

- (a) $\deg y_j \geq \deg x_j$, and that
- (b) $\deg y_j = \deg x_j$ if and only if $y_j = x_j$.

If $G(x_j) \neq x_j$ for some j , by effectiveness as in Definition 5.1, $\deg f(y_1, \dots, y_n) > \deg f$. So $f(y_1, \dots, y_n) \neq_{A^\times} f$. But $f(y_1, \dots, y_n) = G(f) =_{A^\times} f$ by Lemma 4.3(4), a contradiction. Therefore $G(x_j) = x_j$ for all j . As a consequence, $\partial_i(x_j) = 0$ for all i , or $x_j \in \ker \partial$. Since A is generated by x_j 's, $A \subset \ker \partial$. Thus $A \subseteq \text{ML}^H(A[t_1, \dots, t_d])$. It is clear that $A \supseteq \text{ML}^H(A[t_1, \dots, t_d])$, so the assertion.

(2) Let ϕ be the isomorphism from $A \otimes R$ to $B \otimes R$. By Lemma 3.1(b) or [KL, Proposition 3.11]

$$\text{GKdim } B = \text{GKdim}(B \otimes R) - \dim R = \text{GKdim}(A \otimes R) - \text{GKdim } R = \text{GKdim } A < \infty.$$

Let C_A and C_B be the center of A and B respectively. Since R is k -flat, the center of $A \otimes R$ and $B \otimes R$ are $C_A \otimes R$ and $C_B \otimes R$ respectively. By Lemma 4.3(4), ϕ maps $d_w(A \otimes R / C_A \otimes R)$ to $d_w(B \otimes R / C_B \otimes R)$. By Lemma 4.5(2),

$$d_w(A \otimes R / C_A \otimes R) = d_w(A / C_A), \quad \text{and} \quad d_w(B \otimes R / C_B \otimes R) = d_w(B / C_B).$$

This implies that $\phi(d_w(A / C_A)) = d_w(B / C_B) \in B$.

Let f denote $d_w(A / C_A)$. By hypothesis, f is effective. Suppose A is generated by $Y = \bigoplus_{i=1}^n kx_i$ as a k -algebra, and write f as $f(x_1, \dots, x_d)$ as in Definition 5.1. We take the testing algebra to be $T = B \otimes R$. Since T is a domain (since $T \cong A \otimes R$), and R is connected graded, T is an \mathbb{N} -graded domain by setting $\deg b = 0$ for all $b \in B$ and $\deg r = \deg_R r$ for all homogeneous element $r \in R$. In particular, T is an \mathbb{N} -filtered algebra with $F_0 T = B$ such that $\text{gr } T$ is a domain. Now take a testing subset $\{y_1, \dots, y_d\} \subset T$ by setting $y_i = \phi(x_i) \in T$ for $i = 1, \dots, d$. We claim that $y_i \in B$ for all i . If not, there is some i_0 such that y_{i_0} is not in $B = F_0 T$. By the effectiveness of f , $f(y_1, \dots, y_d)$ is either zero or not in $B := F_0 T$. However,

$$f(y_1, \dots, y_d) = f(\phi(x_1), \dots, \phi(x_d)) = \phi(f(x_1, \dots, x_d)) = \phi(f).$$

By the last statement in the previous paragraph, $\phi(f) = d_w(B / C_B)$ is a nonzero element in B , a contradiction. Therefore each $y_i \in B$. This means that ϕ maps x_i to y_i in B . Since A is generated by x_i , the image of A under ϕ is a subalgebra of B . So $\phi^{-1}(B)$ is a subalgebra $A \otimes R$ that contains A as a subalgebra. Note that $\text{GKdim } \phi^{-1}(B) = \text{GKdim } B = \text{GKdim } A$, by the first paragraph of the proof. By Lemma 3.2, $\phi^{-1}(B) = A$. Therefore the image of A under ϕ is exactly B , which implies that $\phi : A \cong B$.

(3) This is a special case of part (2). □

Part (2) of the above theorem shows that A is close to be universally cancellative. Theorem 4.6 is a special case as every dominating element is effective. We are now ready to show Theorem 0.6.

Proof of Theorem 0.6. Since A is finitely generated over its affine center, A has finite GK-dimension. The assertion follows immediately from Theorem 5.2. □

Next we consider some examples studied in [CPWZ1, CPWZ2]. Effectiveness of an element is easy to check sometimes. The following lemma is easy.

Lemma 5.3. *Suppose A is generated by $\{x_1, \dots, x_d\}$ as in Definition 5.1.*

- (1) $f = g_0 x_1 g_1 x_2 \cdots x_{n-1} g_{n-1} x_n g_n$ is effective if $g_i \in A$ are nonzero.
- (2) $f = x_1 x_2 \cdots x_n g$ is effective if $g \in A$ is nonzero.
- (3) If $f = x_1^{b_1} \cdots x_n^{b_n}$, then f is effective if and only if $b_i > 1$ for all i .
- (4) If f is effective, and $g \neq 0$, then fg and gf are effective.
- (5) If f is a polynomial of $x_1^{a_1} \cdots x_n^{a_n}$ for some $a_i \geq 1$, then f is effective.
- (6) Suppose that A is generated by two subalgebras A_1 and A_2 . If f_1 and f_2 are effective elements in A_1 and A_2 respectively, then $f_1 g f_2$ is an effective element in A for any nonzero element $g \in A$.
- (7) If A is generated by $\{x_1, x_2\}$ and $f = g(x_1 x_2 + a x_2 x_1) = h(x_1 x_2 + b x_2 x_1)$ for some scalars $a \neq b$. Then f is effective.
- (8) If d_1, \dots, d_n are positive integers and

$$f = x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n} + \sum_{i_s < d_s} a_{i_1 i_2 \dots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}$$

for some $a_{i_1 i_2 \dots i_n} \in k$, then f is effective.

Next we recall some examples given in [CPWZ1, CPWZ2, CYZ1] that have dominating (and effective) discriminant. In the following examples, we assume that k is a field (and could be a finite field).

Example 5.4. [CYZ1, Theorem 0.1] Let

$$A = k\langle x, y \rangle / (xy - qyx - 1)$$

where $1 \neq q$ is an n th root of unity. Then its center is $C = k[x^n, y^n]$ and A is free over C of rank n^2 . By [CYZ1, Theorem 0.1] the discriminant of $d := d_{n^2}(A/C)$ is of the form

$$d =_{k^\times} x^{n^2(n-1)} y^{n^2(n-1)} + \sum_{j < n^2(n-1)} a_j (xy)^j$$

for some $a_{ij} \in k$. This d is dominating.

Example 5.5. [CPWZ1, Example 5.1] Consider the algebra

$$S(p) := k\langle x, y \rangle / (y^2 x - pxy^2, yx^2 + px^2 y)$$

where $p \in k^\times$. By [AS, (8.11)], $S(p)$ is a noetherian Artin-Schelter regular domain of global dimension 3, which is of type S_2 in the classification given in [AS]. Note that $S(p)$ is 3-Koszul (so not Koszul). Set $A = S(1)$. One can check that the center of A is the commutative polynomial subring $C := k[x^4, y^2, \Omega]$ where $\Omega = (xy)^2 + (yx)^2$. As a C -module, A is free of rank 16. A direct computation shows that

$$d := d_{16}(A/C) =_{k^\times} (x^4)^8 (\Omega^2 + 4x^4 y^4)^8.$$

In the algebra A , d has different presentations

$$(x^4)^8 (\Omega^2 + 4x^4 y^4)^8 = (x^4)^8 (xy + iyx)^{32} = (x^4)^8 (xy - iyx)^{32}$$

where $i^2 = -1$. This d is dominating.

Example 5.6. [CPWZ1, Example 5.6] Suppose 2 is invertible in k . Let A be the algebra

$$k\langle x, y \rangle / (x^2y - yx^2, xy^2 - y^2x, x^6 - y^2).$$

It is isomorphic to the invariant subring $k_{-1}[x_1, x_2]^{S_2}$. Note that A is a connected graded AS Gorenstein algebra with $\deg x = 1$ and $\deg y = 3$. By [CPWZ1, Example 5.6], its discriminant of $d_4(A/C(A))$ is $d := (xy - yx)^4$. Using the relations of A , one has

$$d = ((xy - yx)^2)^2 = ((xy + yx)^2 - 4x^2y^2)^2 = (z^2 - 4x^8)^2 = (z - 2x^4)^2(z + 2x^4)^2$$

where $z = xy + yx$. It is not easy to see whether or not d is dominating (maybe this d is not dominating). Now we show that d is effective.

Consider the trivial filtration on A by taking $F_0A = A$. Pick any two elements, still denoted by x and y in any testing algebra T , and assume that one of them is not in F_0T . Let $z = xy + yx$. Proceed by contradiction and assume that any expression of f is nonzero and in F_0T . So we have that $f(x, y) = (z - 2x^4)^2(z + 2x^4)^2 \neq 0$ and in F_0T . Then both $z - 2x^4$ and $z + 2x^4$ are in F_0T . So x^4 , and whence x , is in F_0T . Thus y must not be in F_0T . By using the fact $f = (xy - yx)^4$ in F_0T , one obtains that $t := xy - yx$ is in F_0T . Thus $z = 2xy - t$ is not in F_0T . This implies that $z \pm 2x^4$ are not in F_0T , which implies that $f = (z - 2x^4)^2(z + 2x^4)^2 \neq 0$ is not in F_0T , a contradiction. Therefore f is effective.

In all above examples, since the discriminant is effective, we have that A is cancellative by Theorem 0.6.

We are now ready to prove Theorem 0.7. We give an expanded version of it.

Theorem 5.7. *Let k be a base commutative ring containing all $p_{ij}^{\pm 1}$ and A be a skew polynomial ring $k_{p_{ij}}[x_1, \dots, x_n]$ where each p_{ij} is a root of unity. Let C be the center of A . The following are equivalent.*

- (1) *The full automorphism group $\text{Aut}(A)$ is affine* [CPWZ1, Definition 2.5].
- (2) *Discriminant $d_w(A/C)$ is dominating where $w = \text{rk}(A/C)$.*
- (3) *C is a subalgebra of $k[x_1^{\alpha_1}, \dots, x_n^{\alpha_n}]$ for some $\alpha_1, \dots, \alpha_n \geq 2$.*
- (4) *Discriminant $d_w(A/C)$ is effective.*
- (5) *x_i divides $d_w(A/C)$ for all $i = 1, \dots, n$.*
- (6) *A is LND^H -rigid.*
- (7) *A is strongly LND^H -rigid.*

If, further, k contains \mathbb{Q} , then the above are also equivalent to

- (8) *A is LND -rigid.*
- (9) *A is strongly LND -rigid.*

Proof. By [CPWZ2, Theorem 3.1], (1), (2) and (3) are equivalent. By [CPWZ2, Theorem 2.11], (2) and (5) are equivalent. By the proof of [CPWZ2, Theorem 2.11], (4) and (5) are equivalent.

(2) \Rightarrow (7): This is Theorem 4.6(1).

(7) \Rightarrow (6): Clear.

(6) \Rightarrow (5): If (5) fails, by [CPWZ2, Theorem 2.11], there is a homogeneous element f of degree at least 2 such that $x_i f = p_{si} f x_i$ for all s (such an element corresponds to an element in T_s (E6.1.1) in the next section). Then

$$g : x_i \rightarrow \begin{cases} x_i & i \neq s \\ x_s + tf & s = i \end{cases}, \quad \text{and} \quad t \rightarrow t$$

defines an algebra automorphism of $A[t]$ over $k[t]$. By Lemma 2.2(3), $g = G_{\partial, t}$ for some nonzero $\partial \in \text{LND}^H(A)$, yielding a contradiction.

Next assume that k contains \mathbb{Q} .

(7) \Rightarrow (9): This follows from the fact the map $\text{LND}^H(A) \rightarrow \text{LND}(A)$ is surjective.

(9) \Rightarrow (8): Obvious.

(8) \Rightarrow (2): [CPWZ2, Theorem 3.1]. \square

6. MAKAR-LIMANOV INVARIANT OF SKEW POLYNOMIAL RINGS

In this section we study $\text{ML}^*(k_{p_{ij}}[x_1, \dots, x_n])$. We start with the following example.

Example 6.1. Suppose $n \geq 3$ is odd and $1 \neq q$ is a root of unity. Let $A = k_q[x_1, \dots, x_n]$ where k contains $q^{\pm 1}$ and \mathbb{Z} . Then $\text{ML}(A) = k$. To see this, we first construct some locally nilpotent derivations. Suppose the order of q is ℓ . If w is odd, let ∂_w be the locally nilpotent derivation of A determined by

$$x_i \mapsto \begin{cases} 0 & \text{if } i \neq w \\ x_1^{\ell-1} x_2^{\ell-1} \cdots \widehat{x_w^{\ell-1}} \cdots x_{n-2}^{\ell-1} x_{n-1}^{\ell-1} x_n^{\ell-1} & \text{if } i = w. \end{cases}$$

If w is even, let ∂_w be the locally nilpotent derivation of A determined by

$$x_i \mapsto \begin{cases} 0 & \text{if } i \neq w \\ x_1 x_2^{\ell-1} x_3 \cdots \widehat{x_w^{\ell-1}} \cdots x_{n-2} x_{n-1}^{\ell-1} x_n & \text{if } i = w. \end{cases}$$

For any $f \in A \setminus k$, there is a w and polynomials f_i of $x_1, \dots, \widehat{x_w}, \dots, x_n$ such that $f = \sum_{i=0}^n f_i x_w^i$ with $f_n \neq 0$ and $n > 0$. Then $\partial_w(f) = \sum_{i=1}^n f_i i \partial_w(x_w) x_w^{i-1} \neq 0$. So $f \notin \text{ML}(A)$ and the assertion follows.

Fixed a parameter set $\{p_{ij} \mid 1 \leq i < j \leq n\}$. For any $1 \leq s \leq n$, let

$$(E6.1.1) \quad T_s = \{(d_1, \dots, \hat{d}_s, \dots, d_n) \in \mathbb{N}^{n-1} \mid \prod_{j=1, j \neq s}^n p_{ij}^{d_j} = p_{is}, \forall i \neq s\}.$$

Similar to the argument in Example 6.1 we have the following result.

Theorem 6.2. Let $A = k_{p_{ij}}[x_1, \dots, x_n]$ where all p_{ij} are roots of unity.

- (1) $\text{ML}^H(A)$ is the subalgebra of A generated by $\{x_s \mid T_s = \emptyset\}$. As a consequence, $\text{ML}^H(A)$ is a skew polynomial ring.
- (2) Suppose that k contain \mathbb{Z} . Then $\text{ML}^I(A)$ is the subalgebra of A generated by $\{x_s \mid T_s = \emptyset\}$. As a consequence, $\text{ML}^I(A)$ is a skew polynomial ring.

Proof. The proofs of (1) and (2) are similar, so we only prove (2). By replacing k by its fraction field, we may assume that k is a field containing \mathbb{Q} . In this case, $\text{ML} = \text{ML}^I$.

Let B be the subalgebra generated by $\{x_s \mid T_s = \emptyset\}$. First we show that $\text{ML}(A) \subset B$. Pick any $f \in A \setminus B$. Then there is a w such that $T_w \neq \emptyset$ and polynomials f_i of $x_1, \dots, \widehat{x_w}, \dots, x_n$ such that $f = \sum_{i=0}^n f_i x_w^i$ with $f_n \neq 0$ and $n > 0$. Since $T_w \neq \emptyset$, pick any $(d_1, \dots, \hat{d}_w, \dots, d_n) \in T_w$. Using this element, we define a locally nilpotent derivation ∂_w as follows:

$$\partial_w : x_i \mapsto \begin{cases} 0 & \text{if } i \neq w \\ x_1^{d_1} x_2^{d_2} \cdots \widehat{x_w} \cdots x_{n-1}^{d_{n-1}} x_n^{d_n} & \text{if } i = w. \end{cases}$$

Then $\partial_w(f) = \sum_{i=1}^n f_i i \partial_w(x_w) x_w^{i-1} \neq 0$. So f is not in $\text{ML}(A)$. This implies that $\text{ML}(A) \subset B$.

To finish the argument, one needs to show that $\partial(B) = 0$ for every $\partial \in \text{LND}(A)$. Or it suffices to show that $\partial(x_s) = 0$ for all s satisfying $T_s = \emptyset$.

So we need to the following claim: if $\partial \in \text{LND}(A)$ and $T_s = \emptyset$, then $\partial(x_s) = 0$. Let g be the automorphism of the $k[t]$ -algebra $A[t]$ of the form $\exp(t\partial)$. By [CPWZ2, Theorem 2.11(2)], $d(A[t]/C) = \prod_{\{s|T_s=\emptyset\}} x_s^{a_s}$ for some $a_s > 0$. Since any automorphism preserves $d(A[t]/C)$, $g(x_s)$ has t -degree 0. This implies that $\partial(x_s) = 0$, as required. \square

7. MOD- p REDUCTION

In this section we introduce a method that deals with the **ZCP** for certain non-PI algebras. We start with a temporary definition.

Definition 7.1. Let A be a k -algebra that is free over k . Fix a k -basis $\{x_i\}_{i \in I}$ of A . Let K be a subring of k . We say a subring $B \subset A$ is a K -order of A if $\{x_i\}_{i \in I}$ is a K -basis of B .

Lemma 7.2. Let K be a commutative domain and A be a K -algebra with a K -basis $\{x_i\}_{i \in I}$. Assume that K is affine over \mathbb{Z} . Suppose that, for any quotient field $F := K/\mathfrak{m}$, $A \otimes_K F$ is LND^H -rigid (respectively, strongly LND^H -rigid). Then A is LND^H -rigid (respectively, strongly LND^H -rigid).

Proof. Let d be a non-negative integer. We only need to show that

$$A \subseteq \text{ML}^H(A[t_1, \dots, t_d])$$

if the above holds when replacing A by $A_F := A \otimes_K F$ for all quotient field $F = K/\mathfrak{m}$. We proceed by contradiction. If $\partial := \{\partial_j\}_{j \geq 0} \in \text{LND}^H(A[t_1, \dots, t_d])$ and $\partial(f) \neq 0$ for some $f \in A$. Then there is some $j \geq 0$ such that $\partial_j(f) = \sum_{i \in I} c_i x_i$ where some c_{i_0} is nonzero. Consider $K_1 = K[c_{i_0}^{-1}]$ and take a quotient field F of K_1 . Then F is finite and F is a quotient field of K as well. Remember that c_{i_0} is invertible in K_1 , whence invertible in F . Write $\partial_F = (\partial \otimes_K F)$. It is clear that ∂_F is in $\text{LND}^H(A_F[t_1, \dots, t_d])$ since G_{t, ∂_F} is the automorphism $G_{t, \partial} \otimes_K F \in \text{Aut}(A_F[t_1, \dots, t_d][t])$. Let f' be the image of f in A_F . Then $(\partial_F)_j(f') = \sum_{i \in I} c'_i x_i \neq 0$ where c'_i are image of c_i in F . This contradicts the fact that $\text{ML}^H(A_F[t_1, \dots, t_d]) = A_F$. Therefore the assertion follows. \square

Definition 7.3. Let A be a K -algebra with a K -basis $\{x_i\}_{i \in I}$. We call $\{x_i\}$ *manageable* if for each $w \geq 0$ and each K -algebra automorphism $G \in \text{Aut}(A[y_1, \dots, y_w])$ there is an affine \mathbb{Z} -subalgebra $K_1 \subset K$ such that $B := \oplus_{i \in I} K_1 x_i$ is a K_1 -order and G is induced from an automorphism of $B[y_1, \dots, y_w]$.

Lemma 7.4. Let K be a commutative domain and A be a finitely generated K -algebra with a manageable K -basis $\{x_i\}_{i \in I}$. If, for every affine \mathbb{Z} -subalgebra $K_1 \subset K$, there is an affine \mathbb{Z} -subalgebra $K_2 \subset K$ containing K_1 such that $B := \oplus_{i \in I} K_2 x_i$ is a K_2 -order of A and that B is LND^H -rigid (respectively, strongly LND^H -rigid), then A is LND^H -rigid (respectively, strongly LND^H -rigid).

Proof. Again, let d be a non-negative integer, we will show that

$$A \subseteq \text{ML}^H(A[t_1, \dots, t_d]).$$

If not, pick $\partial \in \text{LND}^H(A[t_1, \dots, t_d])$ and $f \in A$ such that $\partial(f) \neq 0$. Write, for every $j \geq 0$, $\partial_j(f) = \sum_{i \in I} c_{ji} x_i$ where some $c_{j_0 i_0}$ is nonzero. Consider $G = G_{t, \partial} \in \text{Aut}(A[t_1, \dots, t_d][t])$. Since $\{x_i\}_{i \in I}$ is manageable, $G = H \otimes_{K_2} K$ where K_2 is a finitely generated \mathbb{Z} -subalgebra of K and $H \in \text{Aut}(B[t_1, \dots, t_d][t])$. By the hypothesis, we may assume that K_2 contains all c_{ji} and $\text{ML}^H(B[t_1, \dots, t_d]) = B$. Since $H(b) \equiv b \pmod{t}$ for all $b \in B[t_1, \dots, t_d]$, by Lemma 2.2(3), $H = G_{t, \partial'}$. Since $\text{ML}^H(B[t_1, \dots, t_d]) = B$, $\partial'(f) = 0$, which implies that $\partial(f) = 0$, a contradiction. \square

Here is the main result of this section.

Theorem 7.5. *Let $\Phi := \{A\}$ be a collection of finitely generated algebras over various base commutative domains K . Assume that the following hold.*

- (1) *Each A is finitely generated over K and has a manageable K -basis.*
- (2) *If $A \in \Phi$ where A is a K -algebra and K is an affine \mathbb{Z} -algebra, then $A \otimes_K F$ is in Φ where F is a finite quotient field of K .*
- (3) *If $A \in \Phi$ where A is a K -algebra, then for any affine \mathbb{Z} -subalgebra $K_1 \subset K$, there is an affine \mathbb{Z} -subalgebra $K_2 \subset K$ such that $A = B \otimes_{K_1} K$ for $B \in \Phi$ that is a K_2 -order of A .*
- (4) *If K is a finite field, then A is LND^H -rigid (respectively, strongly LND^H -rigid).*

Then every A in Φ is LND^H -rigid (respectively, strongly LND^H -rigid).

Proof. By Lemma 7.2 and hypothesis (4), A is LND^H -rigid (respectively, strongly LND^H -rigid) if K is affine over \mathbb{Z} . Then by Lemma 7.4 and hypothesis (3), every A in Φ is LND^H -rigid (respectively, strongly LND^H -rigid). \square

Now we are ready to prove Theorem 0.8.

Proof of Theorem 0.8. We now construct the collection Φ as follows: a K -algebra A is in Φ if A is a finite tensor product (over K) of different copies $K_p[x_1, \dots, x_n]$ when n is even (for different values of p), copies of $K\langle x, y \rangle / (x^2 y - y x^2, y^2 x + x y^2)$, and different copies and different copies $K\langle x, y \rangle / (y x - q x y - 1)$ (for different values of q). We require that the base commutative rings K are domains containing the following elements

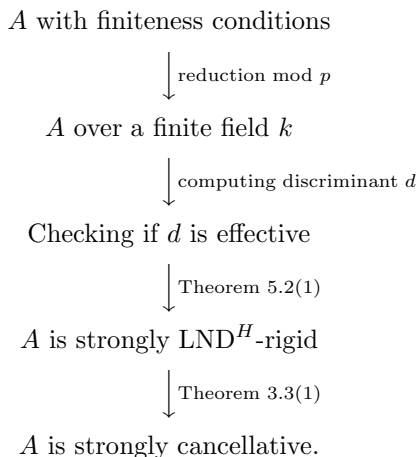
$$(E7.5.1) \quad 2^{-1}, p^{\pm 1}, (p-1)^{-1}, (q-1)^{-1}$$

for different p and q occurring in A . In particular, $p \neq 1$ and $q \neq 1$ and $2 \neq 0$. If K is a finite field, Example 4.7(4) says that A has dominating discriminant. By Theorems 4.6(1) and 5.2(1), A is strongly LND^H -rigid. This verified hypothesis (4) of Theorem 7.5. Hypotheses (1,2,3) of Theorem 7.5 are easy to verify. Therefore every member in Φ is strongly LND^H -rigid.

Since every A is a domain of finite GK-dimension, by Theorem 3.3, A is strongly cancellative. \square

We summarize key steps of solving **ZCP** for noncommutative algebras similar to those in Theorem 0.8 as follows. For an algebra A over a base commutative ring k satisfying certain finiteness conditions, one uses reduction modulo p to reduce the problem in the special case when k is a finite field. When k is a finite field, A is assumed to be PI (which is true for a large class of quantized algebras). Then one can compute the discriminant of A over its center, say $d := d(A/C)$. If one

can verify that d is effective (or dominating), then A is strongly LND^H -rigid by Theorems 4.6(1) and 5.2(1). Finally, by Theorem 3.3(1), A is strongly cancellative. So we have the following diagram.



For algebras of GK-dimension two we should apply Theorem 0.5 directly.

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